International Journal of Theoretical Physics, Vol. 21, Nos. 3/4, 1982

Fluctuations and Dissipation in an Elementary One-Bit Information Storage System

Madhu S. Gupta

Department of Information Engineering, University of Illinois at Chicago Circle, Chicago, Illinois 60680

Received May 6, 1981

A bistable system, capable of storing one bit of information, and exemplified by a particle in a dual-trough potential well, is modeled to determine the effects of a coupling between its information-bearing and the thermal degrees of freedom. Energy dissipation accompanying the change of the stored information is calculated in terms of parameters describing the potential well. The fluctuations cause spontaneous changes in the stored information, and the mean time for this error is also calculated in terms of the well parameters. The results can be used for determining the influence of the well parameters on the performance of the information storage system.

1. OBJECTIVES

The purpose of this paper is (i) to describe a very specific model for a two-state system capable of storing one bit of information, (ii) to calculate the energy dissipation involved in switching the state of the system (i.e., complementing the stored bit of information), and (iii) to calculate the rate of error (defined as the spontaneous switching of system state due to fluctuations), expressed in terms of the mean time for the error in an ensemble of identically prepared systems. These three goals are briefly explained below.

1.1. The Model. The model proposed here for the elementary information storage system is neither new (i.e., many authors have worked with rather similar models), nor unusual (i.e., many kinds of actual physical systems are described by this same model). Hopefully, the system description presented here is complete and precise so that the range of applicability of the results based on this model is known. The following specifications

0020-7748/82/0400-0275$03.00/0

apply to the system:

(a) The system stores one bit of information using only one degree of freedom (labeled x).

(b) Each of the two states of the system (designated by x_0 and x_1), used to represent the stored information, is locally asymptotically stable.

(c) The two states x_0 and x_1 are separated from each other by a finite, static energy barrier, (i.e., the system can undergo a transition from one to the other state by a suitable transfer of energy to the information-bearing degree of freedom). This is indicated in Figure 1.

(d) The system has many other nonessential (i.e., "thermal") degrees of freedom, with the distribution of energy among them corresponding to a constant absolute temperature T.

(e) The thermal degrees of freedom are coupled (i.e., exchange energy) with the information-bearing degree of freedom.

(f) The only mechanism for switching the system state is by "transport over the barrier," and other possibilities, such as tunneling through the barrier, can be disregarded.

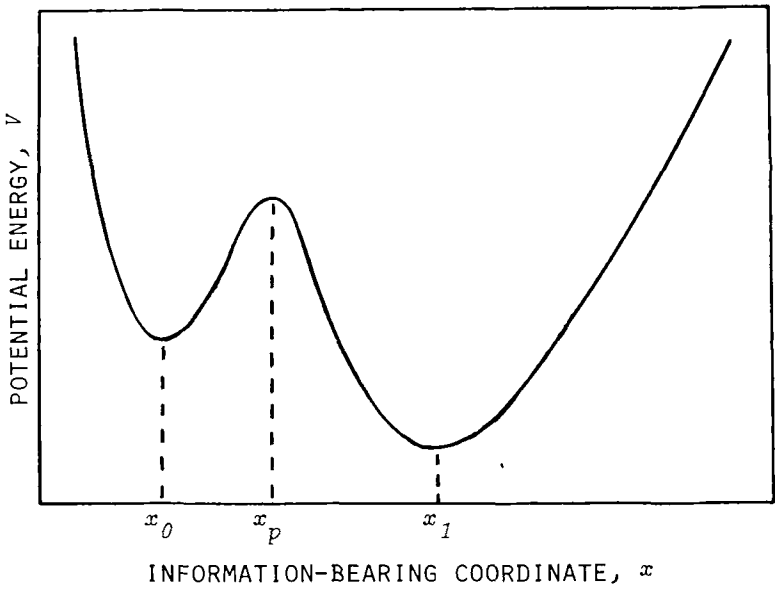

Fig. 1. Variation of potential energy in the information-bearing degree of freedom, as a function of the coordinate representing that degree of freedom, for a bistable information storage system.

1.2. A Canonical Example. The above model specifications are well exemplified by the prototype example of a particle in a one-dimensional, dual-trough potential well with frictional walls, which exhibits the essential features of the class of systems of interest. In this example, the system is replaced by a Newtonian particle, the information-bearing degree of freedom is the particle location along the horizontal x axis, the energy barrier is gravitational, the interaction with thermal degrees of freedom takes place via the frictional walls of the trough at temperature T on which the particle slides, and the two locally stable states are the trough minima. For simplicity, all subsequent work is couched in the language of this example.

1.3. The Dissipation. Dissipation is the transfer of energy from the information-bearing to the thermal degrees of freedom and occurs because the system attempts to relax to the equilibrium state. In the system under consideration, the dissipation of interest is that which occurs when the state of the system is deliberately switched (say, from x_0 to x_1) at a finite rate. As the dissipation depends upon the speed with which the transition is accomplished, the constraint is imposed that the switching time be T_x. A detailed consideration of the dissipation mechanism will be avoided by postulating a phenomenological "frictional coefficient" for the well walls with which the particle interacts. Furthermore, this friction is assumed to be the only source of dissipation; all other possible loss mechanisms (such as inelastic collisions of the particle with the wall in a square well) are disregarded here. Finally, an information processing system may involve dissipative operations other than changing the stored bit of information in a storage system, and these other operations are not examined here.

1.4. The Error Rate. The finite amount of energy in the thermal degrees of freedom, and the coupling between thermal and information-bearing degrees of freedom, manifests itself as spontaneous fluctuation in the energy contained in the information-bearing degree of freedom x. The only consequence of these fluctuations, which is of present interest, is the possibility of an "error," defined as the spontaneous switching of the state of the system. In essence, the particle representative of the system is viewed as performing a random walk, with its first transition past the barrier (i.e., point x_p in Fig. 1) constituting an error.

1.5. Motivation and Scope of Results. As this paper is based entirely on one particular model of an information storage system, the results obtained herein cannot be used (i) to make any universal statements about the tradeoff between fluctuations and dissipation, (ii) to determine any absolute minimum of energy dissipation or any fundamental error rate

limitations applicable to all information storage systems, or (iii) to deduce similar results for an entire information-processing system composed of many elementary storage and logical systems. The results based on the present model are nevertheless of interest because the model contains the essential features of many actual systems in which switching occurs via transport over an energy barrier, and it does not assume a specific functional form for the potential barrier $V(x)$ so that the effect of barrier shape can be studied. As the model employed involves energy dissipation, it might appear to be "suboptimal" compared to a reversible computer element; however, when the switching proceeds deliberately and with a specified rate rather than by Brownian motion, a reversible computer also requires some dissipation.

1.6. Switching Action. The switching operation in the system under consideration requires that some energy be transferred to the information-bearing degree of freedom. This can be accomplished in several ways, for example, (i) by imparting an initial kinetic energy to the particle at x_0 by an impulse, or (ii) by storing an initial potential energy in the particle by drawing it up the barrier wall to the left of x_0, or (iii) by applying a suitable time-varying external field which applies a force on the particle (this is equivalent to modulating the potential barrier by superimposing a second potential). All of these possibilities are admissible in the present model, provided the potential barrier $V(x)$ is properly interpreted when it is time modulated. As the dissipation occurs during intentional switching, which in turn is carried out by time variation of the potential barrier, $V(x)$ should be treated as the instantaneous barrier in the calculation of dissipation. By contrast, when the information is to be retained, the barrier will usually be maintained in some time-invariant state, and if this is the predominant state of the system, spontaneous switching (i.e., error) will occur during such "hold" conditions. The potential barrier $V(x)$ used in the calculation of error rate is therefore the "holding" barrier.

2. CALCULATION OF DISSIPATION

At the phenomenological level, the particle can be described by an equation of motion which accounts for the only two forces acting on the particle: gravitational and frictional. With the mass of particle taken to be unity,

$$\frac{d^2x}{dt^2} \equiv \frac{dv}{dt} = F(x) - \frac{1}{\mu}v \tag{1}$$

where $F(x)$ is the position-dependent external (i.e., gravitational) force, derived from the dual-well potential

$$F(x) = -\frac{d}{dx} V(x) \tag{2}$$

and $(1/\mu)$ is the frictional coefficient, or, equivalently, μ is the mobility of the particle.

The first quantity of interest to be calculated is the energy dissipation accompanying the switching operation, i.e., the transport of the particle from x_0 to x_1. Work done in transporting the particle from x_0 to x_1 is

$$\begin{aligned} W &= \int_{x_0}^{x_1} \left(F(x) - \frac{1}{\mu} v \right) dx \\ &= [V(x_0) - V(x_1)] - \int_{x_0}^{x_1} \frac{1}{\mu} v \, dx \end{aligned} \tag{3}$$

of which only the second part is dissipated. The dissipation is thus found to be

$$E_{\text{diss}} = \int_{x_0}^{x_1} \frac{1}{\mu} v \, dx \tag{4}$$

subject to the switching time constraint:

$$T_s = \int_{x_0}^{x_1} \frac{1}{v} dx \tag{5}$$

The exact amount of dissipation clearly depends on the variation of velocity during switching, which in turn depends on how the energy to be dissipated is supplied to the particle; this is not prespecified in the present model. The simplest possible case is perhaps one for which v is constant, so that

$$T_s = \frac{x_1 - x_0}{v} \tag{6}$$

and

$$E_{\text{diss}} = \frac{(x_1 - x_0)v}{\mu} = \frac{(x_1 - x_0)^2}{\mu T_s} \tag{7}$$

As would be expected intuitively, the dissipation is larger for wider potential wells, smaller mobility (i.e., larger frictional coefficient), and shorter switching time.

3. CALCULATION OF ERROR RATE

Errors in the system are caused by fluctuations, arising from the interaction of the particle with its surroundings (i.e., the walls of the well), maintained at temperature T. The effect of thermal fluctuations can be incorporated in the description of the system by the so-called Langevin method. In this method, the phenomenological equation of motion of the particle is modified by the addition of a random force $r(t)$ to the phenomenological force, in order to account for the random, zero-time-average effect of fluctuations:

$$\frac{dv}{dt} = F(x) - \left(\frac{1}{\mu}\right) v + r(t) \tag{8}$$

As a result of the Langevin force, the position and velocity of the particle along the x coordinate becomes random and can only be described by a probability distribution. Let $f(x, v; t)dx\,dv$ be the probability that the particle position and velocity lie in the intervals x to $x + dx$ and v to $v + dv$, respectively, at time instant t. If the Langevin force $r(t)$ is Markovian, the phase-space probability density $f(x, v; t)$ can be shown to be the solution of the Fokker–Planck equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial v} = \frac{1}{\mu}\frac{\partial}{\partial v}\left(vf + kT\frac{\partial f}{\partial v}\right) \tag{9}$$

As the switching error is defined through the position of the particle, regardless of its velocity, it is more convenient to define the following conditional probability function:

$p(x; t | x_0; 0)\,dx$ = the probability that the particle lies in the interval x to $x + dx$ at the time instant t, given that it was present at x_0 at time instant $t = 0$. This probability function will be abbreviated as $p(x, t)$ in the following, and it can be expressed in terms of $f(x, v; t)$.

The coupling between the particle and the heat bath is strong when the frictional or damping coefficient is large (its reciprocal, the particle mobility μ, is small). Under these conditions, the particle is at local equilibrium at each value of x, and its motion is essentially diffusive. Then the Fokker–

Planck equation in $f(x, v; t)$ leads to a Smoluchowski equation in $p(x, t)$:

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[D\left(\frac{\partial p}{\partial x} + \frac{p}{kT}\frac{dV}{dx} \right) \right] \tag{10}$$

where D represents μkT, and denotes the "diffusion coefficient" for the particle. If D is independent of x, Smoluchowski equation simplifies to the form

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + \frac{D}{kT}\frac{\partial}{\partial x}\left(p\frac{dV}{dx} \right) \tag{11}$$

If the initial state of the system is known as

$$p(x, t) = \delta(x_0) \text{ at } t = 0 \tag{12}$$

then the mean escape time of the particle can be defined as

$$T_0 = 2\int_0^\infty t\frac{\partial}{\partial t}\left[\int_{-\infty}^{x_p} p(x, t)\, dx \right] dt \tag{13}$$

where x_p corresponds to the peak of the potential barrier, as indicated in Fig. 1, and serves as the marker for defining whether an error has occurred or not. The factor 2 arises from the fact that a particle reaching x_p has equal probability of either returning to the first trough, or going on to the second trough. The mean escape time can be evaluated by Weaver's (1979) method in a closed form:

$$T_0 = \frac{1}{D}\int_{x_0}^{x_p} \exp\frac{V(z)}{kT}\left\{ \int_{-\infty}^{z} \exp\left[-\frac{V(y)}{kT} \right] dy \right\} dz \tag{14}$$

This expression assumes the initial position of the particle to be at x_0, which is the most probable position for a system known to be in state "0," since quasiequilibrium within a trough is attained on a very short time scale; in addition, if the well walls are steep, the particle distribution is sharply peaked near x_0. The rate of fluctuation-induced errors is thus found directly in terms of the potential barrier $V(x)$ for the particle.

The result of equation (14) is useful for comparing the error proneness of various possible potential barriers. In addition, the equations (7) and (14) show that both E_{diss} and T_0 are inversely proportional to μ, and therefore there is a tradeoff between dissipation and error rate with respect to the choice of particle mobility.

4. DISCUSSION OF RESULTS

There have been a number of earlier attempts at calculating the dissipation and fluctuation effects in binary devices, and many different results are available in the literature. These differences arise from the use of different models, and therefore the results have different ranges of applicability. In order to place the present results into proper perspective, a small but representative set of earlier models, and their distinguishing features, are summarized here.

(i) A large number of authors have modeled the switching process in a binary device as the gain of one bit of information, which must be paid for by energy dissipation, and have thus arrived at energy dissipation requirements of the order of kT. A typical example of a paper with this line of reasoning is the one by Neyman (1966).

(ii) Stein (1977) related the energy dissipation per logical operation to the maximum tolerable error probability for a system in which two states of the binary information retaining device differ from each other in energy, and this entire energy difference is dissipated in switching. He further calculated the error rate by assuming an equilibrium Boltzmann distribution of energies among the binary devices.

(iii) The paper which comes closest to the approach taken here is that of Landauer and Woo (1973). They take a binary element with a very specific shape for the potential well (namely, a sawtooth), where the switching is carried out by time modulation (more specifically, bifurcation, and reunification) of the potential well. In this model, dissipation occurs in several phases of the operation, each of which is separately calculated, and their results (e.g., their equation 3.9) are the same as equation (7) in the present paper. The fluctuations in their model arise from the particle being caught on the wrong side of the barrier during the bifurcation of the well. Their results are more restrictive than the present ones since they apply to a particular well shape; however, they also consider the case of underdamped wells, where the Smoluchowski equation approximation does not apply.

REFERENCES

Landauer, R., and Woo, J. W. F. (1971). *Journal of Applied Physics*, **42**, 2301.
Neyman, M. S. (1966). *Telecommunications and Radio Engineering*, **21**, 68.
Stein, K.-U. (1977). *IEEE Journal of Solid-State Circuits*, **SC-12**, 527.
Weaver, D. L. (1979). *Physical Rev. B*, **20**, 2558.